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The Asymptotic Expansion of Certain Canonical Integrals

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Integrals of the form $I(\lambda)=\int_{R^2} f(\bar{x})\exp\{i\lambda\phi(\bar{x})\}\,d\bar{x}$, common in diffraction theory, are usually evaluated asymptotically for large λ using the stationary phase technique. At caustic points the classical technique is not applicable. A computation-oriented algorithm is developed for determining the asymptotic expansion of such integrals with $\phi(\bar{x})$ a Thom umbilic, the canonical form appropriate at caustic points for most cases of physical interest.

1. INTRODUCTION

Integrals of the form

$$I(\lambda)=\int_{R^2} f(\bar{x})\exp\{i\lambda\phi(\bar{x})\}\,d\bar{x}, \tag{1}$$

where λ is a large parameter, are common to many branches of physics. In wave propagation, $f(\bar{x})$ is often referred to as an amplitude and $\phi(\bar{x})$ as a phase. Such integrals are usually evaluated asymptotically using the Kelvin stationary phase technique. At caustic (turning) points, i.e., $\bar{x}_0$ such that

$$\nabla\phi(\bar{x}_0)=0,$$

$$\det\left(\frac{\partial^2\phi}{\partial x_i\partial x_j}(\bar{x}_0)\right)=0,$$

the classical stationary phase technique does not apply [1].

A number of techniques have been developed to determine the asymptotic series of such integrals at caustic points. Because at caustic points the phase

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may usually be transformed to a canonical form [2, Chapter 2], it is the canonical integral that is considered. For caustic points where the determinant of the phase (Hessian determinant) vanishes but the matrix $((\partial^2\phi/\partial x_i \partial x_j)(\bar{x}_0))$ is not itself zero, the canonical forms for most cases of physical interest are given by

$$\phi(x,y)=\begin{cases} x^3+y^2 & \text{fold} \quad (2a)\\ x^4+y^2 & \text{cusp} \quad (2b)\\ x^5+y^2 & \text{swallowtail} \quad (2c)\\ x^6+y^2 & \text{butterfly.} \quad (2d)\end{cases}$$

An explicit algorithm for computing the asymptotic series expansion for integrals with phases characterized by such forms appears in [3]. In the cases where the Hessian matrix $((\partial^2\phi/\partial x_i \partial x_j)(\bar{x}_0))$ is itself zero, the canonical forms (for co-dimension $\leqslant 4$) for the phase are given by

$$\phi(x,y)=\begin{cases} x^3-xy^2, & \text{elliptic umbilic} \quad (3a)\\ x^3+y^3 \text{ or } x^3+xy^2, & \text{hyperbolic umbilic} \quad (3b)\\ x^4+xy^2, & \text{parabolic umbilic.} \quad (3c)\end{cases}$$

Determination of an explicit asymptotic series expansion for integrals characterized by such forms is more difficult than in the case above.

The evaluation of such integrals has long been a topic of interest. Recently, in [4], Malgrange showed that integrals characterized by phases more general than in Eqs. (2) or (3), admit to asymptotic expansions of the form

$$I(\lambda)=\sum_{\alpha\beta\gamma} C_{\alpha\beta\gamma}(f)\,\lambda^{\alpha-\beta}(\log\lambda)^{\gamma}.$$

This same result is also attributed to Bernstein [5]. Neither treatment provides a computationally feasible algorithm for the coefficients in the expansion. In [6, 7], Duistermaat outlines an algorithm, introduced by Bleistein [8], for determining the asymptotic series expansion at caustic points for phases of the form (3). The algorithm obtains the asymptotic series in terms of certain generalized Airy integrals, left unevaluated. Apparently, the evaluation of these integrals remains an open problem, further complicated in that some of the required integrals are only defined through regularization [9, Chapter 9]. In [10], Ursell considers a more general problem, namely, double complex integrals whose phase is the uniform unfolding of the elliptic and hyperbolic umbilics, i.e.,

$$\phi_E(x,y)=x^3-xy^2+a_1x^2+a_2x+a_3\,y,$$

$$\phi_H(x,y)=x^3+xy^2+a_1x^2+a_2x+a_3\,y,$$

where a_i are constants. By considering the integrals in complex space, Ursell's algorithm circumvents regularization; but the scope of his treatment precludes the technique from being calculation oriented.

2. Synopsis

In this note we present an algorithm for obtaining the complete asymptotic expansion at the highest order turning point of the canonical integrals whose phase is one of the elliptic, hyperbolic or parabolic umbilics. (The requisite transformations necessary to carry a particular integral to one of these canonical integrals are discussed in [2].) We use the device of Bleistein [9], which is related to the expansions of Malgrange and Bernstein. Our purpose is to present an explicit algorithm suitable for calculations. Analogous to Ursell's treatment, we evade the integral convergence difficulties by replacing the domain of integration R^2 with the disc $\bar{\underline{X}}$ of radius R centered at the origin. For the support of $f(x) \subset$ interior $\bar{\underline{X}}$, the asymptotic series expansion of

$$I(\lambda, \bar{\underline{X}})f = \int_{\bar{\underline{X}}} f(\bar{x}) \exp\{i\lambda\phi(\bar{x})\}\, d\bar{x}$$

is the same as that of $I(\lambda)$. Then the technique of Duistermaat is applicable, except that an integral over the circle $\partial\bar{\underline{X}}$ bounding $\bar{\underline{X}}$ is introduced. The asymptotic expansion of this boundary integral proceeds from the classical stationary phase technique, with the radius R of $\partial\bar{\underline{X}}$ obtainable from Sard's theorem, e.g., [11, Chapter 3].

Thus we consider integral operators of the explicit form

$$I(\lambda, \bar{\underline{X}})f = \iint_{\bar{\underline{X}}} f(x, y) \exp\{i\lambda\phi(x, y)\}\, dx\, dy, \tag{4}$$

where $\phi(x, y)$ is a Thom umbilic, Eqs. (3a)–(c). We assume $f(x, y)$ is analytic in a neighborhood of the stationary point of $\phi(x, y)$ and $\bar{\underline{X}}$ is a smoothly bounded compact region of integration. For brevity (and because it is not handled in Ursell's treatment), we detail only the case where $\phi(x, y)$ is the parabolic umbilic; for comparison, intermediate results for the elliptic and hyperbolic umbilics are presented in the text with final results included in the summary. Further, for expositional clarity, we shall retain the terminology of wave propagation, referring to the integrand as the product of an amplitude and an exponential.

3. The Algorithm

We begin by expanding $f(x,y)$ in Eq. (4) as in Bleistein [8]

$$f(x,y)=\sum_{kl}\alpha_{kl}x^k y^l+\frac{\partial\phi}{\partial x}Af+\frac{\partial\phi}{\partial y}Bf,$$

where the sum over k, l is finite and corresponds to the unfolding of the specific umbilic [Appendix]. A and B are operators, which, along with the constant coefficients α_{kl}, are determined in the Appendix. For the parabolic umbilic

$$f(x,y)=\alpha_{00}+\alpha_{10}x+\alpha_{01}y+\alpha_{20}x^2+\alpha_{02}y^2+(4x^3+y^2)Af+2xyBf. \quad (5)$$

Operating on $f(x,y)$ with

$$I(\lambda,\underline{\bar{X}})=\iint_{\underline{X}}(\ )\exp\{i\lambda\phi(x,y)\}\,dx\,dy$$

one obtains

$$\begin{aligned}I(\lambda,\underline{\bar{X}})f&=\alpha_{00}I_{00}+\alpha_{10}I_{10}+\alpha_{01}I_{01}+\alpha_{20}I_{20}+\alpha_{02}I_{02}\\&\quad+I(\lambda,\underline{\bar{X}})\{(4x^3+y^2)Af+2xyBf\},\end{aligned} \quad (6)$$

where

$$I_{kl}=\iint_{\underline{X}}\exp\{i\lambda\phi(x,y)\}\,x^k y^l\,dx\,dy, \quad (7)$$

with k, l corresponding to the powers of x and y in the umbilic unfolding and where the remainder integral is

$$I(\lambda,\underline{\bar{X}})\left\{\frac{\partial\phi}{\partial x}Af+\frac{\partial\phi}{\partial y}Bf\right\}=\iint_{\underline{X}}\exp\{i\lambda\phi(x,y)\}\left(\frac{\partial\phi}{\partial x}Af+\frac{\partial\phi}{\partial y}Bf\right)dx\,dy. \quad (8)$$

The basic procedure may now be outlined. The I_{kl} integrals and the remainder integral are handled separately. Each I_{kl} integral is transformed to an integral whose integrand is a surface integral, with a classically obtainable asymptotic expansion. Then an asymptotic integration (justified below) determines the asymptotic expansion of each I_{kl} integral. The remainder integral, which yields the higher order terms, is transformed to the sum of a line integral and an integral similar to $I(\lambda,\underline{\bar{X}})\,f(x,y)$, Eq. (4). The line integral has a classically obtainable asymptotic expansion. The other integral is evaluated by repeating the procedure, resulting in an operator formalism for determining the asymptotic series of $I(\lambda, \underline{X}) f(x, y)$ similar to that in [3].

3.1. *The I_{kl} Integrals*

The I_{kl} integrals are evaluated by differentiating Eq. (7) with respect to λ and factoring the resulting integrand into the form $\bar{U} \cdot \nabla g$, i.e.,

$$\frac{dI_{kl}}{d\lambda} = i \iint_{\underline{X}} \exp\{i\lambda\phi(x, y)\}(x^4 + xy^2)\, x^k y^l \, dx\, dy$$

$$= \frac{1}{\lambda} \iint_{\underline{X}} (\nabla \exp\{i\lambda\phi(x, y)\}) \cdot \left(\frac{1}{4} x\hat{\imath} + \frac{3}{8} y\hat{\jmath}\right) x^k y^l \, dx\, dy \tag{9}$$

for the parabolic umbilic. Then using the vector identities

$$\nabla \cdot (h\bar{U}) = h\nabla \cdot \bar{U} + \bar{U} \cdot \nabla h$$

and (10)

$$\int_{\underline{X}} \bar{U} \cdot \nabla h \, ds = \int_{\partial x} h\bar{U} \cdot dl - \int_{\underline{X}} h\nabla \cdot \bar{U}\, ds,$$

with $\partial\underline{X}$ the boundary of $\underline{X}$ and $dl = \hat{\imath}\, dy - \hat{\jmath}\, dx$, Eq. (9) becomes

$$\frac{dI_{kl}}{d\lambda} = \frac{1}{\lambda} \int_{\partial\underline{X}} \exp\{i\lambda\phi(x, y)\}\, x^k y^l \left(\frac{1}{4} x\hat{\imath} + \frac{3}{8} y\hat{\jmath}\right)$$

$$\cdot (\hat{\imath}\, dy - \hat{\jmath}\, dx) - \left(\frac{k+1}{4\lambda} + \frac{3(l+1)}{8\lambda}\right) I_{kl}. \tag{11}$$

Now restricting our attention to where $\underline{X}$ is a disk of radius R, centered at $\bar{0}$, the substitution $x = R \cos\theta$, $y = R \sin\theta$ transforms the boundary integral to

$$I_{\partial\underline{X}} = \frac{R^{2+k+l}}{8\lambda} \int_0^{2\pi} \exp\{i\lambda(R^4 \cos^4\theta + R^3 \cos\theta \sin^2\theta)\}$$

$$\times \cos^{k+2}\theta \sin^l\theta \, d\theta$$

$$+ \frac{3R^{2+k+l}}{8\lambda} \int_0^{2\pi} \exp\{i\lambda(R^4 \cos^4\theta + R^3 \cos\theta \sin^2\theta)\}$$

$$\times \cos^k\theta \sin^{l+2}\theta \, d\theta = \lambda^{-1}\tilde{I}_{1p} + \lambda^{-1}\tilde{I}_{2p}. \tag{12}$$

Thus differential equation (11) becomes

$$I'_{kl} = \lambda^{-1}\tilde{I}_{1p} + \lambda^{-1}\tilde{I}_{2p} - \lambda^{-1}\left(\frac{k+1}{4} + \frac{3(l+1)}{8}\right) I_{kl}; \tag{13a}$$

the analogous equation for the other umbilics is

$$I'_{kl} = \lambda^{-1}\tilde{I} - \lambda^{-1}\left(\frac{2+k+l}{3}\right) I_{kl}. \tag{13b}$$

$$\therefore I_{kl} = C_1\lambda^{-\delta}\int_0^\lambda \tau^{-\rho}\tilde{I}_{1p}(\tau)\,d\tau + C_2\lambda^{-\delta}\int_0^\lambda \tau^{-\rho}\tilde{I}_{2p}(\tau)\,d\tau, \tag{14a}$$

where δ is the coefficient multiplying $\lambda^{-1}I_{kl}$ in Eq. (13a), $\rho = \delta - 1$, $C_1 = \frac{1}{8}R^{2+k+l}$ and $C_2 = \frac{3}{8}R^{2+k+l}$. Correspondingly for Eq. (13b),

$$I_{kl} = \lambda^{-\delta}\int_0^\lambda \tau^{-\rho}\tilde{I}(\tau)\,d\tau. \tag{14b}$$

Integrals (12)—which appear in integrands (14)—may be evaluated asymptotically using the stationary phase technique where the principal contribution to the integral comes from the neighborhood of the stationary point (i.e., θ_0 such that $(d\phi/d\theta)(\theta_0) = 0$, where ϕ is understood to be the transformed ϕ in (11)), i.e.,

TABLE I

Summary of Stationary Points

Umbilic	Transformed ϕ	Stationary points
Elliptic	$(R^3/2)(\cos\theta + \cos 3\theta)$	$\theta_0 = 0, \pi, \pm\cos^{-1}(\pm\sqrt{6}/6)$
Hyperbolic	$R^3\cos\theta$	$\theta_0 = 0, \pi$
Parabolic	$R^4\cos^4\theta + R^3\cos\theta\sin^2\theta$	$\theta_0 = 0, \pi, \pm\cos^{-1}\gamma,$

where

$$\gamma = \sqrt[3]{\frac{1-8R^2}{64R^3} + \frac{\sqrt{4R^2-1}}{16R^2}} + \sqrt[3]{\frac{1-8R^2}{64R^3} - \frac{\sqrt{4R^2-1}}{16R^2}} + \frac{1}{4R},$$

$$R > \frac{1}{2}.$$

For the elliptic and hyperbolic umbilic, the stationary points are all non-degenerate so the usual technique applies; for the parabolic umbilic, most choices of R determine non-degenerate stationary points. (See Table I.)

The procedure begins with a coordinate transformation carrying the exponential argument to a quadratic form and the stationary point to the origin. Next the Cauchy inversion technique is used to express the transformed amplitude in a power series in the transformed coordinate [Appendix]. Then applying the stationary phase theorem of Duistermaat

[6, 7] to the power series determines the asymptotic series of the transformed integrand. Thus, Eq. (12) becomes

$$
\begin{aligned}
I_{\partial \underline{X}} &= \frac{1}{4\lambda} R^{2+k+l} \int \exp\{i\lambda(T_0(R,\theta_0) + T_2(R,\theta_0)\, r^2)\} \\
&\quad \times \frac{\cos^{k+2}(u_{\theta_0}^{-1}(r)+\theta_0) \sin^l(u_{\theta_0}^{-1}(r)+\theta_0)\, dr}{u'(u_{\theta_0}^{-1}(r))} \\
&\quad + \frac{3R^{2+k+l}}{8\lambda} \int \exp\{i\lambda(T_0(R,\theta_0) + T_2(R,\theta_0)\, r^2)\} \\
&\quad \times \frac{\cos^{k}(u_{\theta_0}^{-1}(r)+\theta_0) \sin^{l+2}(u_{\theta_0}^{-1}(r)+\theta_0)\, dr}{u'(u_{\theta_0}^{-1}(r))} \\
&= \frac{1}{4\lambda} R^{2+k+l} \exp\{i\lambda(T_0(R,\theta_0))\} \\
&\quad \times \int \exp\{i\lambda T_2(R,\theta_0)\, r^2\} \sum_{n=0}^{\infty} \kappa_{1n}(R,\theta_0)\, r^n\, dr \\
&\quad + \frac{3R^{2+k+l}}{8\lambda} \exp\{i\lambda(T_0(R,\theta_0))\} \\
&\quad \times \int \exp\{i\lambda T_2(R,\theta_0)\, r^2\} \sum_{n=0}^{\infty} \kappa_{2n}(R,\theta_0)\, r^n\, dr,
\end{aligned}
$$

where $r = u_{\theta_0}(\theta - \theta_0)$, $dr = u'_{\theta_0}(\theta-\theta_0)\, d\theta$, $u_{\theta_0}^{-1}(r) = \theta - \theta_0$, i.e., $u_{\theta_0}(\theta-\theta_0)$ is a dummy transformation (Appendix) carrying $\phi(R, \theta_0)$ to the quadratic form. A summary is presented in Tables II, III and IV.

For expositional clarity, let the integrands $\tau^{-p}\tilde{I}$ in (14) be referred to as $g(\tau)$. The $\tilde{I}$ are analytic and possess asymptotic series (Table IV),

TABLE II

The Transformed Potential

Umbilic	$T_0(R,\theta_0)$	$T_2(R,\theta_0)$
Elliptic	$\frac{R^3(\cos\theta_0 + \cos 3\theta_0)}{2}$	$\frac{-R^3(\cos\theta_0 + 9\cos 3\theta_0)}{4}$
Hyperbolic	$R^3\cos\theta_0$	$\frac{-R^3\cos\theta_0}{2}$
Parabolic	$R^4\cos^4\theta_0 + R^3\cos\theta_0\sin^2\theta_0$	$\frac{1}{2}(12R^4\cos^2\theta_0\sin^2\theta_0 - 4R^4\cos^4\theta_0 + 2R^3\cos\theta_0 - 9R^3\sin^2\theta_0\cos^2\theta_0)$

TABLE III

The Transformed Amplitudes

Umbilic	$\kappa_n(\theta)$
Elliptic	$\kappa_{nE} = \dfrac{1}{4\pi^2} \iint_{c\times c'} \dfrac{\cos^k(t+\theta_0)\sin^l(t+\theta_0)(\cos\theta_0 + 9\cos 3\theta_0)\, d\xi\, dt}{\xi^n t^{n-1}(\xi^2 t^2(\cos\theta_0 + 9\cos 3\theta_0) + 2(\cos(t+\theta_0) + \cos 3(t+\theta_0) - \cos\theta_0 - \cos 3\theta_0))}$
Hyperbolic	$\kappa_{nH} = \dfrac{1}{4\pi^2} \iint_{c\times c'} \dfrac{\cos^k(t+\theta_0)\sin^l(t+\theta_0)\cos\theta_0\, d\xi\, dt}{\xi^n t^{n-1}(\xi^2 t^2 \cos\theta_0 + 2\cos(t+\theta_0) - 2\cos\theta_0)}$
Parabolic	$\hat{\kappa}_{np} = \dfrac{1}{4\pi^2} \iint_{c\times c'} \dfrac{\cos^k(t+\theta_0)\sin^{l+2}(t+\theta_0)(12R^4\cos^2\theta_0\sin^2\theta_0 - 4R^4\cos^4\theta_0 + 2R^3\cos\theta_0 - 9R^3\sin^2\theta_0\cos^2\theta_0)\, d\xi\, dt}{\xi^n t^{n-1}\left[\xi^2 t^2(12R^4\cos^2\theta_0\sin^2\theta_0 - 4R^4\cos^4\theta_0 + 2R^3\cos\theta_0 - 9R^3\sin^2\theta_0\cos^2\theta_0) - 2(R^4\cos^4(t+\theta_0) + R^3\cos(t+\theta_0)\sin^2(t+\theta_0) + R^4\cos^4\theta_0 + R^3\cos\theta_0\sin^2\theta_0)\right]}$
	$\bar{\kappa}_{np} = \dfrac{1}{4\pi^2} \iint_{c\times c'} \dfrac{\cos^{k+2}(t+\theta_0)\sin^l(t+\theta_0)(12R^4\cos^2\theta_0\sin^2\theta_0 - 4R^4\cos^4\theta_0 + 2R^3\cos\theta_0 - 9R^3\sin^2\theta_0\cos^2\theta_0)\, d\xi\, dt}{\xi^n t^{n-1}\left[\xi^2 t^2(12R^4\cos^2\theta_0\sin^2\theta_0 - 4R^4\cos^4\theta_0 + 2R^3\cos\theta_0 - 9R^3\sin^2\theta_0\cos^2\theta_0) - 2(R^4\cos^4(t+\theta_0) + R^3\cos(t+\theta_0)\sin^2(t+\theta_0) + R^4\cos^4\theta_0 + R^3\cos\theta_0\sin^2\theta_0)\right]}$

TABLE IV

The Asymptotic Series Expansions of the Line Integrals (Eq. 12)

Elliptic umbilic	$\frac{R^{2+k+l}}{3}\left(\frac{2\pi}{\lambda}\right)^{1/2} \sum_{j=1}^{6} \exp\{i\lambda E_0(\theta_{0j})\}[E_2(\theta_{0j})]^{-1/2}$ $\times \exp\left\{\frac{i\pi}{4}\operatorname{Sgn} E_2(\theta_{0j})\right\} \sum_{n=0} \frac{i^n}{n!}\frac{(2n)!\,\kappa_{2nE}(\theta_{0j})}{[2E_2(\theta_{0j})]^n}\lambda^{-n},$
Hyperbolic umbilic	$\frac{R^{2+k+l}}{3}\left(\frac{2\pi}{\lambda}\right)^{1/2} \sum_{j=1}^{2} \exp\{i\lambda H_0(\theta_{0j})\}[2H_2(\theta_{0j})]^{-1/2}$ $\times \exp\left\{\frac{i\pi}{4}\operatorname{Sgn} H_2(\theta_{0j})\right\} \sum_{n=0} \frac{i^n}{n!}\frac{(2n)!\,\kappa_{2nH}(\theta_{0j})}{[4H_2(\theta_{0j})]^n}\lambda^{-n},$
Parabolic umbilic	$R^{2+k+l}\left(\frac{2\pi}{\lambda}\right)^{1/2} \sum_{j=1}^{4} \exp\left\{i\left[\lambda T_0(\theta_{0j}) + \frac{\pi}{4}\operatorname{Sgn} T_2(\theta_{0j})\right]\right\}$ $\times\, [2T_2(\theta_{0j})]^{-1/2} \sum_{n=0} \frac{i^n}{n!}\frac{(2n)!}{[4T_2(\theta_{0j})]^n}\left(\frac{1}{4}\hat{\kappa}_{2np}(\theta_{0j}) + \frac{3}{8}\bar{\kappa}_{2np}(\theta_{0j})\right)\lambda^{-n}$

Note. For clarity, we have used E_i, H_i for the appropriate coefficients for the elliptic and hyperbolic umbilics in Table II and Table III.

$\tilde{I}_S \sim \sum_n \beta_n \lambda^{-1/2-n}$. For the parabolic umbilic, over the range of $k, l, \rho \leqslant |\frac{3}{8}|$; for the elliptic and hyperbolic umbilics, $\rho < |\frac{1}{3}|$. Thus $\tau^{-\rho}\tilde{I}_S$ also determines an asymptotic series

$$\tau^{-\rho}\tilde{I}_S \sim \sum_n \beta_n \tau^{-1/2-n-\rho},$$

where the range of $-\frac{1}{2}-\rho$ over the umbilics is $[-\frac{7}{8}, -\frac{1}{8}]$. (Actually, two separate cases arise for τ in $\tau^{-\rho}\tilde{I}$ can have either a negative or positive exponent. The former will be treated explicitly; the latter reduces to the former.)

Consider $\lambda^{-\delta}\int_0^\lambda g(\tau)\,d\tau$, where $g(\tau)=\tau^{-\rho}\tilde{I}(\tau)$. Select an ε, $0<\varepsilon<\lambda$, which may be problem specific, i.e., related to boundary or initial conditions. Then

$$\int_0^\lambda g(\tau)\,d\tau = \int_0^\varepsilon g(\tau)\,d\tau + \int_\varepsilon^\lambda g(\tau)\,d\tau.$$

$\int_0^\varepsilon g(\tau)\,d\tau$ exists as an improper integral and is a constant, c_1.

$$\int_\varepsilon^\lambda g(\tau)\,d\tau = \int_\varepsilon^\infty - \int_\lambda^\infty (g(\tau) - \beta_0 \exp\{ib\tau\}\,\tau^{-\gamma})\,d\tau$$
$$+\beta_0 \int_\varepsilon^\lambda \exp\{ib\tau\}\,\tau^{-\gamma}\,d\tau, \tag{15}$$

where b, γ are the appropriate exponential argument and exponent of τ, respectively, in $\tau^{-\rho}I_S$ (Table IV). The integral over $[\varepsilon, \infty]$ is a constant c_2. The integral over $[\lambda, \infty]$ is asymptotic to the termwise integration of $\tau^{-\rho}I_S - \beta_0 \exp\{ib\tau\}\,\tau^{-\gamma}$ [12, Chapter 1]. A partial integration of the integral over $[\varepsilon, \lambda]$ yields

$$\beta_0 \int_\varepsilon^\lambda \exp\{ib\tau\}\,\tau^{-\gamma}\,d\tau = \frac{\beta_0}{ib}(\exp\{ib\lambda\}\,\lambda^{-\gamma} - \exp\{ib\varepsilon\}\,\varepsilon^{-\gamma}) + \frac{\beta_0\gamma}{ib}\left\{\int_\varepsilon^\infty - \int_\lambda^\infty [\exp\{ib\tau\}\,\tau^{-\gamma-1}]\,d\tau\right\}.$$

The integral over $[\varepsilon, \infty]$ is a constant c_3. The integral over $[\lambda, \infty]$ may be investigated using two partial integrations to obtain (more generally with $m \geqslant 1$)

$$\int_\lambda^\infty \exp\{ib\tau\}\,\tau^{-\gamma-m}\,d\tau = \frac{i}{b}\exp\{ib\lambda\}\,\lambda^{-\gamma-m}\left[1 - \frac{i(m+\gamma)}{b}\lambda^{-1} - i\frac{(m+\gamma)(m+\gamma+1)}{b} \times \int_\lambda^\infty \exp\{ib\tau\}\,\tau^{-\gamma-m-2}\,d\tau\right],$$

i.e.,

$$\int_\lambda^\infty \exp\{ib\tau\}\,\tau^{-\gamma-m}\,d\tau = \frac{i}{b}\exp\{ib\lambda\}\,\lambda^{-\gamma-m} + o(\lambda^{-m-\gamma}).$$

$$\int_\lambda^\infty \exp\{ib\tau\}\,\tau^{-\gamma-m}\,d\tau \sim \sum_{N\geqslant m} \xi_{mN}\lambda^{-N-\gamma}\exp\{ib\lambda\}.$$

Thus for the parabolic umbilic the I_{kl} in (13) becomes

$$\begin{aligned} I_{kl} \sim \lambda^{-\delta}\frac{R^{2+k+l}}{4}&\left\{\bar{C}_1 + \bar{C}_2 - \int_\lambda^\infty (g(\tau) - \bar{\beta}_0 \exp\{ib\tau\}\,\tau^{-\gamma})\,d\tau\right. \\ &\left.+ \frac{\bar{\beta}_0}{ib}(\exp\{ib\lambda\}\,\lambda^{-\gamma} - \exp\{ib\varepsilon\}\,\varepsilon^{-\gamma}) + \frac{\bar{\beta}_0\gamma\bar{C}_3}{ib} - \frac{\bar{\beta}_0\gamma\Gamma_1(\lambda)}{ib}\right\} \\ + \lambda^{-\delta}\frac{3R^{2+k+l}}{8}&\left\{\tilde{C}_1 + \tilde{C}_2 - \int_\lambda^\infty (g(\tau) - \tilde{\beta}_0 \exp\{ib\tau\}\,\tau^{-\gamma})\,d\tau\right. \\ &\left.+ \frac{\tilde{\beta}_0}{ib}(\exp\{ib\lambda\}\,\lambda^{-\gamma} - \exp\{ib\varepsilon\}\,\varepsilon^{-\gamma}) + \frac{\tilde{\beta}_0\gamma\tilde{C}_3}{ib} - \frac{\tilde{\beta}_0\gamma\Gamma_1(\lambda)}{ib}\right\}, \end{aligned} \tag{16a}$$

where $\Gamma_1(\lambda) = \sum_{N \geqslant m} \xi_{mn} \tau^{-n-\gamma} \exp(ib\lambda)$, with a similar result for the elliptic and hyperbolic umbilics

$$\therefore I_{kl} \sim \lambda^{-\delta} \left(C_1 + C_2 - \int_\lambda^\infty (g(\tau) - \beta_0 \exp\{ib\tau\} \tau^{-\gamma})\, d\tau \right.$$
$$\left. + \frac{\beta_0}{ib} (\exp\{ib\lambda\} \lambda^{-\gamma} - \exp\{ib\varepsilon\} \varepsilon^{-\gamma}) + \frac{\beta_0 \gamma C_3}{ib} - \frac{\beta_0 \gamma \Gamma_1(\lambda)}{ib} \right). \tag{16b}$$

In Eq. (16), $\int_\lambda^\infty (g(\tau) - \beta_0 \exp\{ib\tau\} \tau^{-\gamma})\, d\tau$ denotes the term-by-term integration of the asymptotic series of the integral, i.e., it is its own asymptotic series expansion. When the exponent of τ in $\tau^{-\rho} \tilde{I}$ is positive, i.e., $-\rho < 0$, the integral exists properly and the remainder of the analysis is as above.

3.2. *The Remainder Integral.*

A related procedure applies to the remainder integral appearing in Eq. (7). Using the vector identities (10), the remainder integral (given explicitly in Eq. (8)) becomes

$$I(\lambda, X) \left\{ \frac{\partial \varphi}{\partial x} Af + \frac{\partial \varphi}{\partial y} Bf \right\}$$
$$= \frac{1}{i\lambda} \iint_X (\nabla \exp\{i\lambda\varphi(x, y)\}) \cdot (Af(x, y)\, \hat{\imath} + Bf(x, y) \hat{\jmath})\, dx\, dy$$
$$= \frac{1}{i\lambda} \int_{\partial X} \exp\{i\lambda\varphi(x, y)\} (Af(x, y)\, \hat{\imath} + Bf(x, y) \hat{\jmath}) \cdot \overline{dl}$$
$$- \frac{1}{i\lambda} \iint_X \exp\{i\lambda\varphi(x, y)\}\, \nabla \cdot (Af(x, y)\, \hat{\imath} + Bf(x, y) \hat{\jmath})\, dx\, dy, \tag{17}$$

where $\overline{dl} = \hat{\imath}\, dy - \hat{\jmath}\, dx$. In the line integral, using the transformation $x = R \cos\theta$, $y = R \sin\theta$ obtains

$$J(\lambda, X) f = \frac{1}{i\lambda} \sum_{kl} J_{kl}$$
$$= \frac{1}{i\lambda} \sum_{kl} j_{kl}(R) \int_0^{2\pi} \exp\{i\lambda\varphi(R, \theta)\} \sin^k\theta \cos^l\theta\, d\theta. \tag{18}$$

Equation (18) follows by assuming $Af(x, y) + Bf(x, y)$ analytic near $x^2 + y^2 = R^2$. Thus a series $\sum_{kl} j_{kl}(R) \sin^k\theta \cos^l\theta$ and its θ derivatives converge uniformly to $Af(x, y) + Bf(x, y)$ and its θ derivatives. Then from Duistermaat's stationary phase theorem the coefficients of the λ^{-m}'s for the partial sums converge to those for the sum $\sum_{kl} j_{kl}(R) \sin^k \theta \cos^l \theta$, yielding Eq. (18). The J_{kl} are obtained as were the $\tilde{I}$ (Eq. (11)), using stationary phase, and are listed in Table V.

In Table V the sum over j indicates the stationary points (Table I); the definitions of the appropriate coefficients carry over from Tables II and III and the factor of λ^{-1} appearing in Eq. (18) has been included.

To consider the integral over $\underline{X}$ in Eq. (17) we define the operator G

$$Gf = \frac{\partial}{\partial x}(Af) + \frac{\partial}{\partial y}(Bf) \tag{19}$$

leading to

$$\frac{1}{i\lambda} \iint_{\underline{X}} \exp\{i\lambda\phi\} \nabla \cdot (\hat{\imath}Af + \hat{\jmath}Bf)\, dx\, dy = \frac{1}{i\lambda} I(\lambda, \underline{X})\, Gf. \tag{20}$$

Thus Eq. (6) may be expressed as

$$\begin{aligned} I(\lambda, \underline{X}) f = {} & \alpha_{00} I_{00} + \alpha_{10} I_{10} + \alpha_{01} I_{01} + \alpha_{20} I_{20} \\ & + \alpha_{02} I_{02} + J(\lambda, \underline{X}) - \frac{1}{i\lambda} I(\lambda, \underline{X})\, Gf(x, y). \end{aligned} \tag{21}$$

TABLE V

Asymptotic Expansion of the J_{kl} Integrals (Eq. (18))

Umbilic	Asymptotic Expansion of J_{kl}
Elliptic	$j_{kl}(R)\left(\frac{2\pi}{\lambda}\right)^{1/2} \sum_{j=1}^{6} \exp\left\{i\lambda \frac{R^3}{2} E_0(\theta_{0j})\right\} [R^3 E_2(\theta_{0j})]^{-1/2}$ $\times \exp\left\{\frac{i\pi}{4} \operatorname{Sgn} E_2(\theta_{0j})\right\} \sum_{n=0} \frac{i^n (2n)!}{n!} \frac{\kappa_{2nE}(\theta_{0j})\, \lambda^{-n}}{[2R^3 E_2(\theta_{0j})]^n}$,
Hyperbolic	$j_{kl}(R)\left(\frac{2\pi}{\lambda}\right)^{1/2} \sum_{j=1}^{2} \exp\{i\lambda R^3 H_0(\theta_{0j})\} [2R^3 H_2(\theta_{0j})]^{-1/2}$ $\times \exp\left\{\frac{i\pi}{4} \operatorname{Sgn} H_2(\theta_{0j})\right\} \sum_{n=0} \frac{i^n (2n)!}{n!} \frac{\kappa_{2nH}(\theta_{0j})\, \lambda^{-n}}{[4R^3 H_2(\theta_{0j})]^n}$,
Parabolic	$j_{kl}(R)\left(\frac{2\pi}{\lambda}\right)^{1/2} \sum_{j=1}^{4} \exp\{i\lambda P_0(R, \theta_{0j})\} [2P_2(R, \theta_{0j})]^{-1/2}$ $\times \exp\left\{\frac{i\pi}{4} \operatorname{Sgn} P_2(R, \theta_{0j})\right\} \sum_{n=0} \frac{i^n (2n)!}{n!} \left(\frac{1}{4} \hat{\kappa}_{2np}(\theta_{0j}) + \frac{3}{8} \bar{\kappa}_{2np}(\theta_{0j})\right) \lambda^{-n}$.

The higher order terms in the series are obtained by treating $I(\lambda, \underline{\bar{X}})\, Gf(x, y)$ as the integral in Eq. (4). We note that Eq. (21) implicitly defines the operator

$$I(\lambda, \underline{\bar{X}}) = (\tilde{\alpha}_{00} I_{00} + \tilde{\alpha}_{10} I_{10} + \tilde{\alpha}_{01} I_{01} + \tilde{\alpha}_{20} I_{20} + \tilde{\alpha}_{02} I_{02} + J(\lambda, \underline{\bar{X}})) \sum_{N=0} \left(\frac{i}{\lambda}\right)^{N} G^{N}, \tag{22}$$

where the $\tilde{\alpha}_{kl}$ are the operators which assign to $f(x, y) = \sum_{kl} \alpha_{kl} x^k y^l$ the coefficients defined in the Appendix and $I_{kl} = I_{kl}(\lambda)$ are the asymptotic series determined in Subsection 3.1. While the focus has been on the parabolic umbilic, analogous results apply to the elliptic and hyperbolic umbilic.

4. Summary

We have considered integrals of the form

$$I(\lambda, \underline{\bar{X}}) f = \iint_{\underline{\bar{X}}} f(x, y) \exp\{i\lambda\phi(x, y)\}\, dx\, dy,$$

where $\phi(x, y)$ is a Thom umbilic, Eqs. (3a)–(c), and λ a large parameter. Our aim has been to determine an explicit asymptotic series expansion at the stationary points. Summarizing, $I(\lambda, \underline{\bar{X}}) f$ may be expanded as

$$I(\lambda, \underline{\bar{X}}) f = \sum_{kl} \alpha_{kl} I_{kl} + \sum_{mn} J_{mn} - \frac{1}{i\lambda} I(\lambda, \underline{\bar{X}})(Af + Bf). \tag{23}$$

In Eq. (23), the α_{kl} are constants and A and B are operators; both are determined for each umbilic in the Appendix. The asymptotic series of $I(\lambda, \underline{\bar{X}})$ is obtained as the sum of the asymptotic series of each term in Eq. (23). The asymptotic series of the I_{kl} are obtained in Eqs. (16a) and (b). The necessary coefficients for Eqs. (16a) and (b) appear in Tables II–IV. The J_{mn} result from integration of the boundary of $\underline{\bar{X}}$, Eq. (17). The asymptotic series of J_{mn} is given in Table V. The higher order terms in the series for $I(\lambda, \underline{\bar{X}})\ f$ are obtained from $I(\lambda, \underline{\bar{X}})(Af + Bf)$. Specifically, $I(\lambda, \underline{\bar{X}})(Af + Bf)$ is regarded as analogous to $I(\lambda, \underline{\bar{X}})\, f$ and the procedure is repeated, leading to an operator formalism for obtaining the asymptotic series of $I(\lambda, \underline{\bar{X}})\, f$, Eq. (22).

APPENDIX

A.1.

Consider $f(x, y)$ appearing in

$$I(\lambda, \underline{\bar{X}}) f(x, y) = \iint_{\underline{\bar{X}}} f(x, y) \exp\{i\lambda\phi(x, y)\}\, dx\, dy, \tag{A.1}$$

where, as above, $\phi(x, y)$ is a Thom umbilic (Eqs. 3a)–(c), $f(x, y)$ is analytic in a neighborhood of the stationary point of $\phi(x, y)$, λ is a large parameter and $\underline{\bar{X}}$ is a smoothly bounded, compact region of integration. We seek to express $f(x, y)$ in terms of the umbilic unfolding and partial derivations of the umbilic, i.e.,

$$f(x, y) = \sum_{kl} \alpha_{kl} x^k y^l + \frac{\partial\phi}{\partial x} Af + \frac{\partial\phi}{\partial y} Bf,$$

explicitly

$$f(x, y) = \begin{cases} \alpha_{00} + \alpha_{10}x + \alpha_{01}y + \alpha_{20}x^2 & \\ \quad + (3x^2 - y^2) Af - 2xyBf & \text{elliptic} \quad \text{(A.2a)} \\ \alpha_{00} + \alpha_{10}x + \alpha_{01}y + \alpha_{20}x^2 & \\ \quad + (3x^2 + y^2) Af + 2xyBf & \text{hyperbolic} \quad \text{(A.2b)} \\ \alpha_{00} + \alpha_{10}x + \alpha_{01}y + \alpha_{20}x^2 & \\ \quad + \alpha_{02}y^2 + (4x^3 + y^2) Af + 2xyBf & \text{parabolic.} \quad \text{(A.2c)} \end{cases}$$

(We have used the form $x^3 + xy^2$ for the hyperbolic umbilic.) For illustration we consider the parabolic umbilic (A.2c), the others proceed similarly.

Let us first define the operator

$$D_i f(x, y) = \int_0^1 \partial_i f(tx, ty)\, dt.$$

Therefore, D_i (constant) $= 0$, $D_1 x = D_2 y = 1$, $D_1 y = D_2 x = 0$ and

$$\begin{aligned} D_1 D_2 f(x, y) &= \int_0^1 \partial_1 \int_0^1 \partial_2 f(stx, sty)\, ds\, dt \\ &= \int_0^1 \partial_2 \int_0^1 \partial_1 f(stx, sty)\, dt\, ds = D_2 D_1 f(x, y). \end{aligned}$$

Thus

$$D_i f = D_i f(\bar{o}) + x_i D_i^2 f + x_j D_i D_j f \qquad \text{and}$$
$$D_i^2 f = D_i^2 f(\bar{o}) + x_i D_i^3 f + x_j D_j D_i^2 f. \tag{A.3}$$

Expanding $f(x, y)$ in terms of the D_i

$$f(x,y) = f(\bar{o}) + xD_1 f + yD_2 f = f(\bar{o}) + xD_1 f(\bar{o}) + yD_2 f(\bar{o})$$
$$+ x^2 D_1^2 f + y^2 D_2^2 f + 2xyD_1 D_2 f. \tag{A.4}$$

Because we seek to express Eq. (A.4) in the form of Eq. (A.2c), the equality

$$x^2 D_1^2 f + y^2 D_2^2 f + 2xy = \alpha_{20} x^2 + \alpha_{02} y^2 + (4x^3 + y^2) Af + 2xyD_1 D_2 f \tag{A.5}$$

must hold, requiring the determination of α_{20}, α_{02} and the operators A and B. First we expand $x^2 D_1^2 f$ and $y^2 D_2 f$ and regroup to obtain

$$x^2 D_1^2 f = x^2 D_1^2 f(\bar{o}) + x^3 D_1^3 f + x^2 y D_2 D_1^2 f$$
$$= x^2 D_1^2 f(\bar{o}) + (4x^3 + y^2) \tfrac{1}{4} D_1^3 f - \tfrac{1}{4} y^2 D_1^3 f + x^2 y D_2 D_1^2 f, \tag{A.6}$$
$$y^2 D_2^2 f = y^2 D_2^2 f(\bar{o}) + y^3 D_2^3 f + xy^2 D_1 D_2^2 f$$
$$= y^2 D_2^2 f(\bar{o}) + (4x^3 + y^2) y D_2^3 f - 4x^3 y D_2^3 f + xy^2 D_1 D_2^2 f. \tag{A.7}$$

Combining Eqs. (A.5) and (A.6) allows us to express the left hand side of Eq. (A.4) as

$$x^2 D_1^2 f + y^2 D_2^2 f + 2xyD_1 D_2 f$$
$$= x^2 D_1^2 f(\bar{o}) + y^2 D_2^2 f(\bar{o}) + (4x^3 + y^2)\left(\frac{1}{4} D_1^3 f + yD_2^3 f\right) \tag{A.8}$$
$$+ 2xy \left(D_1 D_2 f + \frac{x}{2} D_2 D_1^2 f - 2x^2 D_2^3 f + \frac{y}{2} D_1 D_2^2 f\right) - \frac{1}{4} y^2 D_1^3 f.$$

Similarly, expanding $-\frac{1}{4} y^2 D_1^3 f$ in terms of the partials of $x^4 + xy^2$

$$-\frac{1}{4} y^2 D_1^3 f = -\frac{1}{4} y^2 D_1^3 f(\bar{o}) - \frac{1}{4} (4x^3 + y^2) y D_2 D_1^3 f$$
$$+ 2xy \left(\frac{x}{2} D_2 D_1^3 f - \frac{y}{8} D_1^4 f\right). \tag{A.9}$$

Combining Eqs. (A.7, (A.8) and the second equality in (A.3) obtains for $f(x, y)$

$$
\begin{aligned}
f(x,y) = {} & f(\bar{o}) + xD_1 f(\bar{o}) + yD_2 f(\bar{o}) + x^2 D_1^2 f(\bar{o}) + y^2 \Big(D_1^2 f(\bar{o}) \\
& - \frac{1}{4} D_1^3 f(\bar{o}) \Big) + (4x^3 + y^2)\Big(\frac{1}{4} D_1^3 f + yD_2^3 f - \frac{1}{4} yD_2 D_1^3 f\Big) \\
& + 2xy \Big(D_1 D_2 f + \frac{x}{2} D_2 D_1^2 f - 2x^2 D_2^3 f + \frac{y}{2} D_1 D_2^2 f \\
& + \frac{x^2}{2} D_2 D_1^2 f - \frac{y}{8} D_1^4 f \Big).
\end{aligned} \tag{A.10}
$$

Comparing Eqs. (A.10) and (A.2c) determines the coefficients α_{kl} and the operators A and B. The other umbilics proceed similarly with the results given in Table A.1.

A.2.

Here we detail the technique for obtaining the series coefficients of the transformed amplitude (Table III). Let c and $\tilde{c}$ be neighborhoods of z_0 and ω_0, respectively. If $g(z)$ is an analytic function on c, then $g[f^{-1}(\omega)]$ is analytic at ω_0 and has the Taylor expansion

$$
g[f^{-1}(\omega)] = \sum_{n=0} \left[\frac{1}{2\pi i} \int_c \frac{g(z) f'(z)\, dz}{(f(z) - \omega_0)^{n+1}} \right] (\omega - \omega_0)^{n+1}, \tag{A.11}
$$

valid for all $\omega \in \tilde{c}$. If

$$
g(r) = \frac{\cos^k(u_{\theta_0}^{-1}(r) + \theta_0) \sin^l(u_{\theta_0}^{-1}(r) + \theta_0)}{u'(u_{\theta_0}^{-1}(r))}, \tag{A.12}
$$

i.e., a typical amplitude in the transformed integral, the Cauchy inversion theorem may be used to determine the coefficients κ_n in the series $g(r) = \sum_n \kappa_n r^n$. We illustrate the technique for one integral occurring in the treatment of the parabolic umbilic, the other cases proceed analogously. For notational clarity, we shall let z_0 be the origin.

From Eq. (A.10), with $t = \theta - \theta_0$,

$$
\begin{aligned}
& \frac{\cos^k(u_{\theta_0}^{-1}(r) + \theta_0) \sin^l(u_{\theta_0}^{-1}(r) + \theta_0)}{u'(u_{\theta_0}^{-1}(r))} \\
& \quad = \frac{1}{2\pi i} \int_{0 \in c} \frac{\cos^k(t + \theta_0) \sin^l(t + \theta_0)\, u'_{\theta_0}(t)}{u'_{\theta_0}(t)(u_{\theta_0}(t) - r)}\, dt
\end{aligned} \tag{A.13}
$$

$$
\begin{aligned}
\therefore\ & \frac{\cos^k(u_{\theta_0}^{-1}(r) + \theta_0) \sin^l(u_{\theta_0}^{-1}(r) + \theta_0)}{u'_{\theta_0}(u_{\theta_0}^{-1}(r))} \\
& \quad = \frac{1}{2\pi i} \int_{0 \in c} \frac{\cos^k(t + \theta_0) \sin^l(t + \theta_0)\, u'_{\theta_0}(t)\, dt}{u'_{\theta_0}(t)(1 - r/u_{\theta_0}(t))\, u_{\theta_0}(t)} \\
& \quad = \frac{1}{2\pi i} \sum_n r^n \int_{0 \in c} \frac{\cos^k(t + \theta_0) \sin^l(t + \theta)}{(u_{\theta_0}(t))^{n+1}}\, dt,
\end{aligned} \tag{A.14}
$$

TABLE A.1

The Expansion Coefficients

	α_{00}	α_{10}	α_{01}	α_{20}	α_{02}
Elliptic umbilic	$f(\bar{o})$	$D_1 f(\bar{o})$	$D_2 f(\bar{o})$	$D_1^2 f(\bar{o}) + 3D_2^2 f(\bar{o})$	—
	$Af = -D_2^2 f + \frac{x}{3} D_1^3 f + x D_1 D_2^2 f$			$Bf = -\left(D_1 D_2 f + \frac{3x}{2} D_2^3 f + \frac{y}{2} D_1 D_2^2 f + \frac{y}{6} D_1^3 f + \frac{x}{2} D_2 D_1^2 f\right)$	
Hyperbolic umbilic	$f(\bar{o})$	$D_1 f(\bar{o})$	$D_2 f(\bar{o})$	$D_1^2 f(\bar{o}) - 3D_2^2 f(\bar{o})$	—
	$Af = D_2^2 f + \frac{x}{3} D_1^3 f - x D_1 D_2^2 f$			$Bf = D_1 D_2 f - \frac{3x}{2} D_2^3 f + \frac{y}{2} D_1 D_2^2 f - \frac{y}{6} D_1^3 f + \frac{x}{2} D_2 D_1^2 f$	
Parabolic umbilic	$f(\bar{o})$	$D_1 f(\bar{o})$	$D_2 f(\bar{o})$	$D_1^2 f(\bar{o}) - \frac{1}{4} D_2^3 f(\bar{o})$	$D_2^2 f(\bar{o})$
	$Af = \frac{1}{4} D_1^3 f + y D_2^3 f - \frac{1}{4} y D_2 D_1^3 f$			$Bf = D_1 D_2 f + \frac{x}{2} D_2 D_1^2 f - 2x^2 D_2^3 f + \frac{y}{2} D_1 D_2^2 f + \frac{x^2}{2} D_2 D_1^3 f - \frac{y}{8} D_1^4 f$	

which follows from $u_{r_0}(t)$ having no zeroes on c, consequently a value of z inside c can always be chosen so that $|u_{\theta_0}(z)/u_{\theta_0}(t)| < 1$, when t is on c.

$$\therefore \frac{\cos^k (u_{\theta_0}^{-1}(r) + \theta_0) \sin^l (u_{\theta_0}^{-1}(r) + \theta_0)}{u'(u_{\theta_0}^{-1}(r))} = \sum_{n=0}^{\infty} \kappa_n r^n,$$

where

$$\kappa_n(\theta_0) = \frac{1}{2\pi i} \int_{0 \in c} \frac{\cos^k (t + \theta_0) \sin^l (t + \theta_0)}{(u_{\theta_0}(t))^{n+1}} \, dt. \tag{A.15}$$

A computational simplification arises from the variable change

$$v_{\theta_0}(t) = u_{\theta_0}^2(t)/t^2$$

for then from the Cauchy inversion formula

$$-v_{\theta_0}(t)^{-(n+1)/2} = \frac{1}{2\pi i} \int_{v_{\theta_0}(0) \in c'} \frac{\omega^{-(n+1)/2}}{\omega - v_{\theta_0}(t)} \, dt,$$

where c' is a contour enclosing $v_{\theta_0}(0)$. Let $\omega = \xi^2$, then

$$v_{\theta_0}(t)^{-(n+1)/2} = \frac{1}{\pi i} \int_{v_{\theta_0}(o) \in c'} \frac{d\xi}{\xi^n(\xi^2 - v_{\theta_0}(t))}.$$

Thus, from (A.14)

$$\kappa_n(\theta_0) = -\frac{1}{2\pi^2} \int_{0 \in c'} \frac{\cos^k (t + \theta_0) \sin^l (t + \theta_0)}{t^{n+1}} \left\{ \int_{c'} \frac{d\xi}{\xi^n(\xi^2 - v_{\theta_0}(t))} \right\} dt.$$

Now substituting for $v_{\theta_0}(t)$ in terms of $u_{\theta_0}(t)$, e.g., for the parabolic umbilic

$$u_{\theta_0}^2(t) = 2 \frac{\begin{array}{c} R^4 \cos^4(\theta_0 + t) \\ + R^3 \cos(\theta_0 + t) \sin^2(\theta_0 + t) - R^4 \cos^4 \theta_0 - R^3 \cos \theta_0 \sin^2 \theta_0 \end{array}}{\begin{array}{c} 12R^4 \cos^2 \theta_0 \sin^2 \theta_0 - 4R^4 \cos^4 \theta_0 + 2R^3 \cos \theta_0 \\ - 9R^3 \sin^2 \theta_0 \cos^2 \theta_0 \end{array}}.$$

Thus for one coefficient from Table III

$$\kappa_{1p_n}(R, \theta_0)$$

$$= \frac{1}{4\pi^2} \iint_{c \times c'} \frac{\begin{array}{c} \cos^{k+2}(t + \theta_0) \sin^l(t + \theta_0)(12R^4 \cos^2 \theta_0 \sin^2 \theta_0 - 4R^4 \cos^4 \theta_0 \\ + 2R^3 \cos \theta_0 - 9R^3 \sin^2 \theta_0 \cos^2 \theta_0) \, d\xi \, dt \end{array}}{\begin{array}{c} \xi^n t^{n-1} [\xi^2 t^2 (12R^4 \cos^2 \theta_0 \sin^2 \theta_0 - 4R^4 \cos^4 \theta_0 + 2R^3 \cos \theta_0 \\ - 9R^3 \sin^2 \theta_0 \cos^2 \theta_0) - 2(R^4 \cos^4(t + \theta_0) \\ + R^3 \cos(t + \theta_0) \sin^2(t + \theta_0) + R^4 \cos^4 \theta_0 + R^3 \cos \theta_0 \sin^2 \theta_0)] \end{array}}.$$

Note that it is not necessary to find the coordinate transformation u_{θ_0} to determine the asymptotic series.

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